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Multiplicity with a Thrust Cut

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Abstract

We evaluate the multiplicity of hadrons in the e^+e^- -annihilation at a given thrust T in the modified leading-log approximation, including $O(\sqrt{\alpha_s})$ corrections. The calculation is done at a large value of $\tau = 1 - T$ by the use of the factorisation which takes place in the one-particle-inclusive cross section at a given τ . At a small τ , a different type of factorisation takes place, which also enable us to evaluate the multiplicity. Two approaches are compared numerically. Measuring this quantity near $\tau = 1/3$, we can determine the multiplicity ratio between a gluon-jet and a quark-jet.

1 Introduction

The Modified Leading-Log Approximation (MLLA) [1][2] with the Local Parton-Hadron Duality (LPHD)[3] has been successful in describing the majority of the e^+e^- -annihilation events (see, for example, [4][5][6]). The shower Monte Carlo programmes based on it [7] are widely used for the simulation of experiments. Nevertheless, our experimental knowledge of the multiple hadroproduction in *gluon jets* is still poor, or even apparently in contradiction with the theoretical predictions. The contradiction has been felt particularly in the multiplicity ratio between a gluon-jet and a quark-jet.

In earlier time, some argued that the multiplicity ratio would be identical to the ratio between the colour charges of a gluon and a quark [8]. Nowadays, this expectation is justified in the following way: According to LPHD, the average number (multiplicity) of hadrons produced in a hard process is proportional to the number of gluons emitted perturbatively from the hard partons. Because of the infrared singularity of the emission amplitude, the majority of the gluons are soft (much less energetic than the hard parton). At the leading order in MLLA, the strength of the soft-gluon radiation is proportional to the colour charge of the hard parton ($C_A = 3$ for a gluon, $C_F = 4/3$ for a quark) with which the jet is associated. Thus the multiplicity ratio between a gluon-jet and a quark-jet is expected to be close to $C_A/C_F = 9/4$ at high energies.

The ratio has been measured using three-jet events of the e^+e^- -annihilation, identifying one of the jets as a gluon-jet. The ratio was found much lower than the above expectation C_A/C_F . For example, OPAL collaboration gave [9]

$$r = \frac{\langle n \rangle_{\text{g-jet}}}{\langle n \rangle_{\text{q-jet}}} = 1.267 \pm 0.043(\text{stat.}) \pm 0.055(\text{syst.}) \quad (1)$$

(including neutral particles) and

$$r_{\text{CH}} = \frac{\langle n_{\text{CH}} \rangle_{\text{g-jet}}}{\langle n_{\text{CH}} \rangle_{\text{q-jet}}} = 1.326 \pm 0.054(\text{stat.}) \pm 0.073(\text{syst.}) \quad (2)$$

(charged particles only). (For earlier experiments, see [10].)

The next-to-leading order correction in MLLA ($O(\sqrt{\alpha_s}) = O(1/\sqrt{\ln(W^2/\Lambda_{QCD}^2)})$, W : the total energy) reduces the leading order prediction $r = 9/4$ by about 10 percent [11]. The correction is far too small to explain the discrepancy from the experimental observation. (The next-to-next order correction does not change the situation [12].)

It should be noted, however, that the multiplicity in a gluon-jet $\langle n \rangle_{\text{g-jet}}$, which was calculated in MLLA, is not necessarily identical to the quantity measured in the experiments. In the theoretical calculation, it is defined as a half of the total multiplicity from

the two hard gluons created by a gauge invariant gluon source (for example, $F_{\mu\nu}^2$, where $F_{\mu\nu}$ is the field strength of gluon).

In the experiments, on the other hand, one selects three-jet events (defined one way or another) and compares the number of hadrons in each angular region. At current energies, however, a hard parton does not necessarily produce a well-collimated jet, and it may cause biases, depending on how the measurement is made: If one does not impose the condition that each jet be isolated (i.e. well-collimated), the contribution from each hard parton may mix up one another¹. Thus the difference in the multiplicity due to the difference in the colour charge of the respective hard parton may be reduced in the comparison of the multiplicity in the respective angular region. If one selects the events with well-collimated jets, on the other hand, one is essentially comparing the multiplicity in the direction of the respective hard parton (in a narrow angular cone around the jet axis). Now the stronger radiation of gluons from a hard gluon, which is responsible for a larger multiplicity in a gluon jet, also causes a wider angular distribution of hadrons, owing to the larger recoil from the *multiple* gluon emission from the hard gluon [6] (see the angular distribution of hadrons measured in [9]). The multiplicity counted in the direction of the hard gluon, therefore, is not as large as C_A/C_F times the multiplicity counted in the direction of the hard quark or antiquark (see also [2]).

Though the magnitude of the biases is in general hard to estimate theoretically, we may examine it using the MLLA-based shower Monte Carlo simulation of the experiments. The simulations done by the experimentalists (in [9] and [10]) in fact gave reduced multiplicity ratio. It suggests, at least, that the observed smaller ratio does not necessarily contradict QCD itself.

In any case, a bias free measurement of the multiplicity ratio is not easy, because a gauge-invariant two-gluon source is hard to prepare².

In this article, we analyse the multiplicity at a given value of thrust, where thrust T is defined, in the c.m. frame, by

$$T = \max \left\{ \frac{\sum_i |\vec{P}_i \cdot \vec{n}|}{\sum_i |\vec{P}_i|} \right\} \quad (\vec{n}^2 = 1) \quad (3)$$

(the direction of the three-vector \vec{n} is chosen to maximise the rhs). $T = 1$ would imply that all the particles are parallel (or antiparallel) to \vec{n} .

¹ Strictly speaking, we cannot tell, in a gauge invariant way, which particle is emitted from which hard parton, except in the high-energy limit where only collinear particles are produced.

²The process $\gamma\gamma \rightarrow gg$ may be used for this purpose [13].

If T is far from one, it implies that a hard gluon is emitted. The emitted hard gluon causes an increase of the multiplicity. At high energies, where most of the events with large $\tau = 1 - T (< 1/3)$ are well-collimated 3-jet events, the multiplicity increase can be identified with the contribution from the multiplicity in a gluon-jet. In this way, we can avoid the experimental bias caused by the angular restriction, and we can compare the theoretical calculations with the experimental data without ambiguities.

In Sect.2, we evaluate the multiplicity at a large τ at the next-to-leading order of MLLA. At this order, a hard gluon adds the multiplicity independently of the quark- and the antiquark-jets. The multiplicity increase is thus close to $C_A/(2C_F)$ times the multiplicity from the $q\bar{q}$ -pair, which is identical to the multiplicity $\langle n \rangle_{e^+e^-}$ (without specifying the thrust value) at the same c.m. energy as the $q\bar{q}$ -pair.

The approximation, however, is not accurate if the energy is not high enough or τ is not large enough. In either case, the gluon jet is not well-isolated, and higher order corrections cause substantial modifications.

In Sect.3, we evaluate the multiplicity at a small τ , where no hard gluons are emitted. At the leading order, the multiple soft-gluon emission from the $q\bar{q}$ -pair determines the value of $\tau (\ll 1)$ [14][15]. Because the soft gluons are emitted *independently* from the hard $q\bar{q}$ -pair, the one-particle-inclusive amplitude, which gives the multiplicity, factorises from $d\sigma/d\tau$, and the resummation to all orders can easily be done.

In this approximation, we do not separately evaluate the multiplicity from a gluon-jet and a quark-jet. When we apply the approximation to a larger value of τ , however, we expect two distinctive contributions, one identified as the multiplicity from the gluon-jet and the other from the quark- and antiquark-jets. The approximation is not justified near $\tau = 1/3$, because it assumes that the gluons are emitted independently. The error, however, would not be large in the intermediate τ region. We therefore expect that the prediction in this region is close to the next-to-leading order prediction obtained in Sect.2, if the energy is high enough.

We evaluate the multiplicity in the small τ region making use of the detailed experimental data of the thrust distribution at 58GeV [20]. The multiplicity evaluated in this approximation in the intermediate τ region ($\tau \approx 0.1$ -0.2) turns out to be lower than the prediction in Sect.2. This is because the multiplicity from the hard gluon is not well-separated from the multiplicity from the $q\bar{q}$ -pair. It suggests that we may have to go to higher energies in order to be able to determine the multiplicity ratio from the multiplicity measured at τ in the intermediate region.

2 Multiplicity at large τ (Next-to-Leading Order)

The one-particle-inclusive (1PI) cross section counts the number of particles at a given momentum. When we integrate it over the momentum, and divide it by the cross section (which counts the number of the events), we obtain the number of particles per event. The multiplicity at given $\tau = 1 - T$ is obtained by dividing the integrated 1PI cross section at given τ by the cross section $d\sigma/d\tau$.

When τ is large (but $< 1/3$), a hard gluon is emitted. At high energies, the multiplicity is given as the sum of the two distinctive contributions:

- (a) The registered particle originates from the hard $q\bar{q}$ -pair; or
- (b) it originates from the hard gluon.

In either case, the registered particle is mostly produced near the direction of the respective hard parton.

At the leading order, the multiplicity at a large τ is simply $1 + C_A/(2C_F)$ times the total multiplicity $\langle n \rangle_{e^+e^-}$ at the same energy. The additional multiplicity proportional to $C_A/(2C_F)$ is due to (b) (the multiplicity in a gluon jet).

We first evaluate the next-to-leading order corrections ($O(\sqrt{\alpha_s})$) to (a). Let P_1 (P_2) be the momentum of the hard quark (antiquark), P_3 the momentum of the hard gluon, and k the momentum of the gluon to which the registered hadron belongs (Fig.1a). Suppose that the hard gluon is emitted on the quark side (in the c.m. frame).

At the next-to-leading order, only the soft and/or collinear emission contributes to the multiplicity. We can therefore assume $2P_1 \cdot P_3 \gg m^2$, where $m^2 = 2P_1 \cdot k$ if the k -gluon is emitted on the quark side, and $m^2 = 2P_2 \cdot k$ if it is emitted on the antiquark side. The value of thrust is determined by $2P_1 \cdot P_3$:

$$1 - T = \tau = \frac{2P_1 \cdot P_3}{W^2} + O\left(\frac{m^2}{W^2}\right) , \quad (4)$$

$$2P_1 \cdot P_3 \gg m^2 .$$

Owing to the strong inequality, the cross section for the emission of the two gluons factorises into the cross section for the hard-gluon emission ($d\sigma/d\tau$) times soft- and/or collinear-gluon emission amplitude. Therefore, when we divide the 1PI cross section (illustrated in Fig.1a) by $d\sigma/d\tau$, we obtain

$$\langle n \rangle_{q\bar{q}} = \int_0^{\tau W^{2/2}} \frac{dm^2}{m^2} \int_{m^2/W'^2}^1 dx_1 \frac{2C_F}{\pi} \left(\frac{1}{x_1} - 1 + \frac{x_1}{2} \right) \alpha_s(x_1 m^2) M_g(x_1 m^2) , \quad (5)$$

$$W'^2 = 2P_1 \cdot P_2 ,$$

where $M_g(Q^2)$ is the multiplicity from a gluon emitted at the transverse momentum Q , and the integration kernel represents the probability of the emission of a gluon at the longitudinal momentum fraction x_1 (cf. (10) below). The invariant mass of the $q\bar{q}$ -pair W' depends on the momentum of the hard gluon P_3 . Later, we shall integrate the rhs of (5) over P_3 with the weight of the hard-gluon emission probability.

At the next-to-leading order, MLLA gives the multiplicity function $M_g(Q^2)$ in the form [1][16]

$$M_g(Q^2) = C \left(\ln(Q^2/\Lambda^2) \right)^{\gamma_1} \exp \left[2\gamma_0 \sqrt{\ln(Q^2/\Lambda_{QCD}^2)} \right] , \quad (6)$$

where

$$\begin{aligned} \gamma_0 &= \sqrt{\frac{C_A}{2\pi b_0}} , \\ \gamma_1 &= -\frac{1}{4} - \frac{N_f}{6\pi b_0} \left(1 - \frac{C_F}{C_A} \right) , \\ b_0 &= \frac{11C_A - 2N_f}{12\pi} \end{aligned}$$

(N_f : the number of the active quark flavours). The normalisation constant C in (6) is not determined in the perturbation theory, and has to be fixed by comparing the predictions with the experimental data. The effective QCD coupling $\alpha_s(Q^2)$ becomes large as Q approaches the QCD mass scale Λ_{QCD} , and the interaction becomes nonperturbative. The emission of the strongly interacting gluons at low Q is responsible for the bound state formation, and does not contribute to the multiplicity. $M_g(Q^2)$, which represents the number of perturbatively emitted gluons, therefore, should be set zero for $Q^2 < Q_0^2$ in the integration on the rhs of (5), where the infrared cutoff Q_0 is of the order of Λ_{QCD} . Q_0 is an unknown parameter from the point of view of the perturbation theory. The dependence on Q_0 , however, is absorbed in the normalisation constant C of the multiplicity function M_g : i.e. the ambiguity disappears in the physically observable multiplicities owing to the renormalisation of M_g .

The upper bound of m^2 on the rhs of (5) ($m^2 < \tau W^2/2$) is an ambiguous quantity in our approximation. We have assumed at the accuracy of the next-to-leading order that $\tau W^2 \approx 2P_1 \cdot P_3 \gg m^2$. It implies that the upper bound of m^2 can be anywhere below τW^2 (if not much smaller than it), and that the dependence on the upper bound is indeed at the next-to-next order. This fact can easily be checked by the explicit evaluation of the rhs of (5).

The integration on the rhs of (5) gives at the next-to-leading order

$$\begin{aligned}
\langle n \rangle_{q\bar{q}} &= \frac{4C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(\frac{13}{4} - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} M_g(\tau W^2/2) \\
&\quad - \frac{2C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(\frac{11}{2} - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} M_g \left(\frac{(\tau W^2/2)^2}{W'^2} \right) \\
&= \frac{2C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} M_g(W'^2). \quad (7)
\end{aligned}$$

Let us next evaluate the other contribution (b), in which the registered hadron originates from the hard gluon. In this case, the multiplicity function M_g is attached to the hard-gluon line (momentum P_3) (Fig.1b). Other gluons may be emitted from the $q\bar{q}$ -pair, but they have to be soft and/or collinear gluons at the next-to-leading order in MLLA. For the unregistered soft and/or collinear emission, we should note the following:

- The soft and/or collinear emission does not change the value of thrust.
- The soft and/or collinear emission amplitude factorises.

It implies that the singularity of the real emission amplitude is cancelled by the singularity of the virtual correction. Namely, the same cancellation that occurred in the case of the 1PI cross section without specifying the thrust value takes place also in this case. We can therefore neglect the emission of the unregistered gluons at the next-to-leading order.

Suppose that the event consists in only three (on-shell) hard particles: a quark at a momentum P_1 , an antiquark at P_2 , and a gluon at P_3 . The $O(\alpha_s)$ cross section for the single gluon emission is given by

$$\begin{aligned}
d\sigma &= \sigma_0 8\pi C_F \alpha_s \left\{ \frac{2P_1 \cdot P_3}{(2P_2 \cdot P_3)W^2} + \frac{2P_2 \cdot P_3}{(2P_1 \cdot P_3)W^2} + \frac{2(2P_1 \cdot P_2)}{(2P_1 \cdot P_3)(2P_2 \cdot P_3)} \right\} \frac{d^3P_3}{(2\pi)^3 2P_3}, \quad (8) \\
W^2 &= (P_1 + P_2 + P_3)^2,
\end{aligned}$$

where σ_0 is the lowest order cross section without any gluons emitted ($\gamma^* \rightarrow q\bar{q}$).

We assume for the moment that the antiquark is more energetic than the quark (in the c.m. frame) ($P_2^0 > P_1^0$). We take the direction of the antiquark as the negative z -axis.

The direction of the thrust axis (\vec{n} in (3)) is identical to the direction of the most energetic parton. We therefore have to distinguish the two cases.

(i) If the antiquark is more energetic than the gluon ($P_2^0 > P_3^0$), the z -direction is identical to the thrust axis. We then have

$$2P_1 \cdot P_3 = \tau W^2,$$

$$\begin{aligned}
2P_2 \cdot P_3 &= x(1 - \tau)W^2, \\
2P_1 \cdot P_2 &= (1 - x)(1 - \tau)W^2.
\end{aligned} \tag{9}$$

where $x = P_3^+ / (P_1 + P_3)^+$ ($P_i^+ = (P_i^0 + P_i^3) / \sqrt{2}$).

The differential cross section for the process $\gamma^* \rightarrow q\bar{q}g$ is in this case

$$\begin{aligned}
\frac{d\sigma}{d\tau dx} &= \sigma_0 \frac{C_F \alpha_s}{2\pi} \left\{ \frac{2P_1 \cdot P_3}{2P_2 \cdot P_3} + \frac{2P_2 \cdot P_3}{2P_1 \cdot P_3} + \frac{2(2P_1 \cdot P_2)W^2}{(2P_1 \cdot P_3)(2P_2 \cdot P_3)} \right\} \\
&= \sigma_0 \frac{C_F \alpha_s}{2\pi} \left\{ \frac{\tau}{x(1 - \tau)} - \frac{x(1 - \tau)}{\tau} + \frac{2(1 - x)}{\tau x} \right\} \\
&\equiv \sigma_0 \frac{C_F \alpha_s}{2\pi} f_1(\tau, x).
\end{aligned} \tag{10}$$

The condition $P_2^0 > P_3^0$ implies $2P_1 \cdot P_3 < 2P_1 \cdot P_2$: namely,

$$x < \frac{1 - 2\tau}{1 - \tau}, \tag{11}$$

while $P_2^0 > P_1^0$ implies $2P_1 \cdot P_3 < 2P_2 \cdot P_3$, or

$$\frac{\tau}{1 - \tau} < x. \tag{12}$$

(ii) If $P_2^0 < P_3^0$, the direction of the hard gluon is identical to the thrust axis. We then have

$$\begin{aligned}
2P_1 \cdot P_3 &= \frac{1 - x - \tau}{1 - x} W^2 \\
2P_2 \cdot P_3 &= \frac{x}{1 - x} \tau W^2, \\
2P_1 \cdot P_2 &= \tau W^2,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\frac{d\sigma}{d\tau dx} &= \frac{\sigma_0}{1 - x} \frac{C_F \alpha_s}{2\pi} \left\{ \frac{2P_1 \cdot P_3}{2P_2 \cdot P_3} + \frac{2P_2 \cdot P_3}{2P_1 \cdot P_3} + \frac{2(2P_1 \cdot P_2)W^2}{(2P_1 \cdot P_3)(2P_2 \cdot P_3)} \right\} \\
&= \frac{\sigma_0}{1 - x} \frac{C_F \alpha_s}{2\pi} \left\{ \frac{x\tau}{1 - x - \tau} + \frac{1 - x - \tau}{x\tau} + 2 \frac{(1 - x)^2}{x(1 - x - \tau)} \right\} \\
&\equiv \sigma_0 \frac{C_F \alpha_s}{2\pi} f_2(\tau, x)
\end{aligned} \tag{14}$$

The condition $P_2^0 < P_3^0$ implies $2P_1 \cdot P_3 > 2P_1 \cdot P_2$: namely,

$$x < \frac{1 - 2\tau}{1 - \tau}, \tag{15}$$

while $P_2^0 > P_1^0$ now implies

$$\frac{1-\tau}{1+\tau} < x \quad (16)$$

Finally, the case $P_1^0 > P_2^0$ gives the same cross section (now the direction of the quark is taken as the negative z -direction), and thus simply doubles the contribution.

When a gluon is emitted *at a large angle*, the angular ordering of the succeeding soft-gluon emission is not exact at the next-to-leading order. Accordingly, the multiplicity from the P_3 -gluon is not simply identical to $M_g(Q^2)$, where Q^2 is the transverse component square of P_3 . Instead, it is given by $M_g(2P_1 \cdot P_3) + M_g(2P_2 \cdot P_3) - M_g(2P_1 \cdot P_2)$ ³. The multiplicity from the hard gluon $\langle n \rangle_g$ at a given thrust value ($0 \ll \tau < 1/3$) is then obtained by averaging it with the cross sections (10) and (14) over the range of x given by (11),(12) and (15),(16) respectively:

$$\begin{aligned} \langle n \rangle_g &= \int dx \{M_g(2P_1 \cdot P_3) + M_g(2P_2 \cdot P_3) - M_g(2P_1 \cdot P_2)\} \frac{d\sigma/(dx d\tau)}{d\sigma/d\tau} \\ &= \frac{1}{A(\tau)} \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx \left\{ M_g(\tau W^2) + M_g(x(1-\tau)W^2) - M_g((1-x)(1-\tau)W^2) \right\} \\ &\quad \times f_1(\tau, x) \\ &+ \frac{1}{A(\tau)} \int_{\frac{1-\tau}{1+\tau}}^{\frac{1-2\tau}{1-\tau}} dx \left\{ M_g\left(\frac{1-x-\tau}{1-x}W^2\right) + M_g\left(\frac{x}{1-x}\tau W^2\right) - M_g(\tau W^2) \right\} \\ &\quad \times f_2(\tau, x) , \end{aligned} \quad (17)$$

where

$$\begin{aligned} A(\tau) &= F_1(\tau) + F_2(\tau) , \\ F_1(\tau) &= \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_1(\tau, x) \\ &= \frac{3\tau^2 + 8\tau - 3}{2\tau(1-\tau)} + \frac{\tau^2 - 2\tau + 2}{\tau(1-\tau)} \ln \frac{1-2\tau}{\tau} , \\ F_2(\tau) &= \int_{\frac{1-\tau}{1+\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_2(\tau, x) \\ &= \frac{1+\tau^2}{1-\tau} \left\{ \ln \frac{(1-\tau)^2}{\tau(1+\tau)} - \frac{1}{\tau} \ln \frac{(1-\tau)^2}{(1-2\tau)(1+\tau)} \right\} + 2 \ln \frac{1+\tau}{2(1-\tau)} . \end{aligned} \quad (18)$$

f_1 and f_2 are defined in (10) and (14).

³ In order for this expression to be correct, P_3 has to be hard

Let us now go back to $\langle n \rangle_{q\bar{q}}$. Averaging the rhs of (7) over the hard-gluon momentum, we obtain

$$\begin{aligned}
\langle n \rangle_{q\bar{q}} &= \frac{2C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} \int dx M(W'^2) \frac{d\sigma/(dx d\tau)}{d\sigma/d\tau} \\
&= \frac{2C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} \frac{1}{A(\tau)} \\
&\times \left\{ \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx M_g((1-x)(1-\tau)W^2) f_1(\tau, x) + \int_{\frac{1-\tau}{1+\tau}}^{\frac{1-2\tau}{1-\tau}} dx M_g(\tau W^2) f_2(\tau, x) \right\}.
\end{aligned} \tag{19}$$

The multiplicity at a given τ is the sum of the two contributions:

$$\langle n \rangle_\tau = \langle n \rangle_{q\bar{q}} + \langle n \rangle_g \tag{20}$$

The thrust-dependent multiplicity ratio $R(\tau)$ is defined by

$$R(\tau) = \frac{\langle n \rangle_g}{\langle n \rangle_q}, \tag{21}$$

where

$$\langle n \rangle_q \equiv \frac{1}{2} \langle n \rangle_{q\bar{q}}. \tag{22}$$

The expressions (17) and (19) can be expanded around $\tau = 1/3$. At $O(\sqrt{\alpha_s})$, we obtain

$$\langle n \rangle_g = \left\{ 1 + \frac{\gamma_0}{\sqrt{\ln(W^2/\Lambda_{QCD}^2)}} \frac{B_1(\tau)}{A(\tau)} \right\} M_g\left(\frac{W^2}{3}\right), \tag{23}$$

$$\begin{aligned}
\langle n \rangle_{q\bar{q}} &= \frac{2C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} \\
&\times \left\{ 1 + \frac{\gamma_0}{\sqrt{\ln(W^2/\Lambda_{QCD}^2)}} \frac{B_2(\tau)}{A(\tau)} \right\} M_g\left(\frac{W^2}{3}\right),
\end{aligned} \tag{24}$$

$$\begin{aligned}
B_1(\tau) &= \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_1(\tau, x) \ln \frac{3\tau x}{1-x} \\
&+ \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_2(\tau, x) \ln \frac{3x(1-x-\tau)}{(1-x)^2},
\end{aligned} \tag{25}$$

$$\begin{aligned}
B_2(\tau) &= \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_1(\tau, x) \ln(3(1-\tau)(1-x)) \\
&\quad + \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_2(\tau, x) \ln(3\tau)
\end{aligned} \tag{26}$$

The ratio $R(\tau)$ is thus

$$\begin{aligned}
R(\tau) &= \frac{C_A}{C_F} \left\{ 1 + \frac{\gamma_0}{\sqrt{\ln(W^2/\Lambda_{QCD}^2)}} \frac{B_1(\tau)}{A(\tau)} \right\} \left\{ 1 + \frac{\gamma_0}{\sqrt{\ln(W^2/\Lambda_{QCD}^2)}} \frac{B_2(\tau)}{A(\tau)} \right\}^{-1} \\
&\quad \times \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\}^{-1}
\end{aligned} \tag{27}$$

We can easily perform the integral on the rhs of (25) and (26) to obtain

$$\begin{aligned}
B_1(\tau) &= \ln(3\tau)F_1(\tau) - F_3(\tau) + \frac{1}{2}F_5(\tau) + \ln 3 F_2(\tau) \\
&\quad - F_4(\tau) + F_6(\tau) - F_7(\tau) \quad ,
\end{aligned} \tag{28}$$

$$B_2(\tau) = \ln(3(1-\tau)) F_1(\tau) - \frac{1}{2}F_5(\tau) + \ln(3\tau)F_2(\tau) \quad , \tag{29}$$

where

$$\begin{aligned}
F_3(\tau) &= \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_1(\tau, x) \ln \frac{1}{x} \\
&= \frac{1}{\tau(1-\tau)} \left\{ \frac{1}{2}(1-2\tau)(3+2\tau) \ln \frac{1-2\tau}{1-\tau} + \frac{\tau}{2}(\tau-4) \ln \frac{\tau}{1-\tau} \right. \\
&\quad \left. + \frac{1}{4}(3\tau-1)(7+\tau) - \frac{1}{2}(\tau^2-2\tau+2) \ln \frac{(1-2\tau)\tau}{(1-\tau)^2} \ln \frac{1-2\tau}{\tau} \right\}
\end{aligned}$$

$$\begin{aligned}
F_4(\tau) &= \int_{\frac{1-\tau}{1+\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_2(\tau, x) \ln \frac{1}{x} \\
&= -\frac{1+\tau^2}{1-\tau} \left\{ \text{Sp} \left(\frac{\tau^2}{(1-\tau)^2} \right) - \text{Sp} \left(\frac{\tau}{1+\tau} \right) \right\} + 2 \left\{ \text{Sp} \left(\frac{\tau}{1-\tau} \right) - \text{Sp} \left(\frac{2\tau}{1+\tau} \right) \right\} \\
&\quad + \frac{1+\tau^2}{1-\tau} \ln(1-\tau) \ln \frac{\tau(1+\tau)}{(1-\tau)^2} - \frac{1+\tau^2}{2\tau(1-\tau)} \ln \frac{1-2\tau}{1+\tau} \ln \frac{(1-2\tau)(1+\tau)}{(1-\tau)^2}
\end{aligned}$$

$$\begin{aligned}
F_5(\tau) &= \int_{\frac{\tau}{1-\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_1(\tau, x) \ln \frac{1}{(1-x)^2} \\
&= -\frac{2+3\tau}{1-\tau} \ln \frac{\tau}{1-\tau} + \frac{3-6\tau}{\tau(1-\tau)} \ln \frac{1-2\tau}{1-\tau} - \frac{(1-3\tau)(5+3\tau)}{2\tau(1-\tau)} \\
&\quad + \frac{2(\tau^2-2\tau+2)}{\tau(1-\tau)} \left\{ \text{Sp} \left(\frac{1-2\tau}{1-\tau} \right) - \text{Sp} \left(\frac{\tau}{1-\tau} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
F_6(\tau) &= \int_{\frac{1-\tau}{1+\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_2(\tau, x) \ln \frac{1}{(1-x)^2} \\
&= 2 \frac{1+\tau^2}{1-\tau} \left\{ \text{Sp}(\tau) - \text{Sp}\left(\frac{1-\tau}{2}\right) + \frac{1}{2} \ln \frac{2(1-\tau)}{1+\tau} \ln \frac{1-\tau^2}{2} + \ln \tau \ln \frac{\tau(1+\tau)}{(1-\tau)^2} \right\} \\
&\quad - 2 \ln \frac{2\tau^2}{1-\tau^2} \ln \frac{1+\tau}{2(1-\tau)} + \frac{2(1+\tau^2)}{\tau(1-\tau)} \left\{ \text{Sp}\left(\frac{1-2\tau}{1-\tau}\right) - \text{Sp}\left(\frac{1-\tau}{1+\tau}\right) \right\} \\
F_7(\tau) &= \int_{\frac{1-\tau}{1+\tau}}^{\frac{1-2\tau}{1-\tau}} dx f_2(\tau, x) \ln \frac{1}{1-x-\tau} \\
&= \frac{1+\tau^2}{2(1-\tau)} \ln \frac{\tau^3}{1+\tau} \ln \frac{\tau(1+\tau)}{(1-\tau)^2} + 2 \left\{ \text{Sp}\left(\frac{1+\tau}{2}\right) - \text{Sp}(1-\tau) \right\} \\
&\quad + \ln \frac{1-\tau^2}{2\tau^2} \ln \frac{1+\tau}{2(1-\tau)} + \frac{1+\tau^2}{\tau(1-\tau)} \left\{ \text{Sp}\left(\frac{1-2\tau}{(1-\tau)^2}\right) - \text{Sp}\left(\frac{1}{1+\tau}\right) \right\} \\
&\quad - \frac{1+\tau^2}{\tau(1-\tau)} \ln(1-\tau) \ln \frac{(1-2\tau)(1+\tau)}{(1-\tau)^2} . \tag{30}
\end{aligned}$$

$\text{Sp}(x)$ is the Spence function defined by

$$\text{Sp}(x) = \int_0^x \frac{dx}{x} \ln \frac{1}{1-x} . \tag{31}$$

In the limit $\tau \rightarrow 1/3$, we find that $B_i(\tau)/A(\tau) \rightarrow 0$, ($i = 1, 2$), and

$$R(1/3) = r , \tag{32}$$

where r is the conventional multiplicity ratio between a quark-jet and a gluon-jet calculated at the next-to-leading order [11]:

$$r = \frac{C_A}{C_F} \left\{ 1 - \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} \tag{33}$$

We show in Fig.2 the prediction for $\langle n \rangle_\tau$ divided by $\langle n \rangle_{e^+e^-}$ at $W=58\text{GeV}$. $\langle n \rangle_{e^+e^-}$ is given by

$$\begin{aligned}
\langle n \rangle_{e^+e^-} &= \int_0^{W^2} \frac{dm^2}{m^2} \int_{m^2/W^2}^1 dx_1 \frac{2C_F}{\pi} \left(\frac{1}{x_1} - 1 + \frac{x_1}{2} \right) \alpha_s(x_1 m^2) M_g(x_1 m^2) \\
&= \frac{2C_F}{C_A} \left\{ 1 + \frac{\gamma_0}{3\sqrt{\ln W^2/\Lambda_{QCD}^2}} \left(1 - \frac{N_f}{C_A} \left(2\frac{C_F}{C_A} - 1 \right) \right) \right\} M_g(W^2) . \tag{34}
\end{aligned}$$

In Table 1, the prediction at 58GeV is compared numerically with that at 91GeV. We find that the differences are negligibly small.

In Fig.2, we also show $R(\tau)$ at $W=58\text{GeV}$. It reduces appreciably as τ decreases, while change in $\langle n \rangle_\tau$ is more moderate. As the emitted gluon becomes harder (at larger τ), it carries away a portion of energy from the $q\bar{q}$ -pair, and the increase in $\langle n \rangle_g$ is partially compensated by the decrease in $\langle n \rangle_{q\bar{q}}$ in their sum $\langle n \rangle_\tau$ (recoil effect).

The multiplicity ratio $R(\tau)$ is in general *not* a directly measurable quantity. What we measure without ambiguity is not $\langle n \rangle_g$ or $\langle n \rangle_q$ itself, but their sum $\langle n \rangle_\tau$. However, $R(1/3) = r$ can be determined experimentally in the following way: First, we measure $\langle n \rangle_\tau$ at $\tau = 1/3$. (In practice, because of the lack of statistics near $\tau = 1/3$, we measure $\langle n \rangle_\tau$ for $\tau < 1/3$, and extrapolate it to $\tau = 1/3$, assuming a smooth dependence on τ .) Now we note that $\langle n \rangle_{q\bar{q}}$ at $\tau = 1/3$ is identical to $\langle n \rangle_{e^+e^-}$ at the c.m. energy $W/\sqrt{3}$, which is the c.m. energy of the $q\bar{q}$ pair (compare (7) with (34)). We can therefore use the experimentally measured $\langle n \rangle_{e^+e^-}$ at the c.m. energy $W/\sqrt{3}$ for $\langle n \rangle_{q\bar{q}}$ at $\tau = 1/3$. Then $\langle n \rangle_g$ is defined as $\langle n \rangle_{\tau=1/3}$ minus $\langle n \rangle_{q\bar{q}}$, and $R(1/3)$ is obtained by (21).

It should be noted that the above result (23) and (24) cannot be applied to $\tau > 1/3$, because the $O(\alpha_s)$ cross section vanishes for $\tau > 1/3$.

3 Small τ region

So far, we have analysed the large τ region, assuming that the event consists in three well-separated jets (i.e. the invariant mass of each jet is much smaller than W). When τ is small, multiple soft-gluon emission determines the value of τ [14], and it is no more possible to isolate a single gluon jet. In this case, however, the soft gluons can be regarded as emitted independently from the initial hard $q\bar{q}$ -pair, and the soft-gluon emission amplitude factorises (as is the case with soft-photon emission in QED). Making use of the factorisation, we can evaluate the multiplicity at a given τ without dividing it into the contributions from isolated jets.

Let us first remind ourselves that if all the emitted gluons are soft τ can be expressed as the sum of the contribution of each gluon [14]:

$$\tau \approx \sum_i m_i^2, \quad m_i^2 = \min\{2P_1 \cdot k_i, 2P_2 \cdot k_i\}, \quad (35)$$

where k_i is the momentum of each soft gluon. The emission amplitude of n soft gluons is factorised into the n identical amplitudes of one-gluon emission (Eikonal Approximation),

and we obtain the thrust distribution $F(\tau)$ in the form of a series

$$F(\tau) = \frac{1}{\sigma} \frac{d\sigma}{d\tau} = \Delta(W^2, Q_0^2) \left\{ \Gamma(\tau W^2) + \frac{1}{2!} \int_0^\tau d\tau_1 \Gamma(\tau_1 W^2) \Gamma((1 - \tau_1) W^2) + \dots \right\} \quad (\tau \neq 0) , \quad (36)$$

where $\Gamma(\tau W^2)$ is the factorised amplitude for the single soft-gluon emission (cf. (10))

$$\Gamma(\tau W^2) = \frac{2C_F}{\pi} \frac{1}{\tau} \int_\tau^1 \frac{dx}{x} \alpha_s(x\tau W^2) \theta(x\tau W^2 - Q_0^2) , \quad (37)$$

and Δ is the normalisation factor:

$$\Delta(W^2, Q_0^2) = \exp \left[- \int_0^{1/3} d\tau \Gamma(\tau W^2) \right] . \quad (38)$$

Q_0 is the lower bound of the transverse momentum of the emitted gluon, below which the soft gluon does not contribute to the particle proliferation.

The multiplicity at given τ is equal to the 1PI cross section at given τ integrated over the whole phase space and divided by $d\sigma/d\tau$. The integrated 1PI cross section at a given τ (divided by σ) is obtained by substituting the multiplicity $M_g(x\tau W^2)$ for one of the Γ s in each term of the series on the rhs of (36), thereby substituting $1/(j-1)!$ for the Bose factor $1/j!$ in the j -th term:

$$\begin{aligned} F(\tau) < n >_\tau &= \frac{1}{\sigma} \frac{d\sigma}{d\tau} < n >_\tau \\ &= \Delta(W^2, Q_0^2) \left\{ M'(\tau W^2) + \int_0^\tau \frac{d\tau_1}{\tau_1} M'(\tau_1 W^2) \Gamma((1 - \tau_1) W^2) \right. \\ &\quad \left. + \frac{1}{2!} \int_0^\tau \frac{d\tau_1}{\tau_1} \int_0^{\tau - \tau_1} d\tau_2 M'(\tau_2 W^2) \Gamma(\tau_2 W^2) \Gamma((1 - \tau_1 - \tau_2) W^2) + \dots \right\} \\ &= \int_0^\tau \frac{d\tau_1}{\tau_1} 2M'(\tau_1 W^2) \left\{ \delta(\tau_1 - \tau) + \Gamma((\tau - \tau_1) W^2) \right. \\ &\quad \left. + \frac{1}{2!} \int_0^{\tau - \tau_1} d\tau_2 \Gamma(\tau_2 W^2) \Gamma((1 - \tau_1 - \tau_2) W^2) + \dots \right\} , \end{aligned} \quad (39)$$

where

$$M'(\tau W^2) = \frac{2C_F}{\pi} \int_\tau^1 \frac{dx}{x} \alpha_s(x\tau W^2) M_g(x\tau W^2) . \quad (40)$$

Comparing the rhs of (39) with (36), we obtain

$$< n >_\tau = \frac{\int_0^\tau \frac{d\tau_1}{\tau_1} M'(\tau_1 W^2) F(\tau - \tau_1)}{F(\tau)} . \quad (41)$$

In (37) and (40), we included only the leading order term. The next-to-leading order corrections can be included by substituting

$$\begin{aligned}\Gamma(\tau W^2) &= \frac{1}{\tau} \int_{\tau}^1 dx \left\{ C_F \frac{\alpha_s(x\tau W^2)}{\pi} \left(\frac{1}{x} - 1 + \frac{x}{2} \right) + a^{(2)} \left(\frac{\alpha_s(x\tau W^2)}{\pi} \right)^2 \frac{1}{x} \right\} \\ &\quad \times \theta(x\tau W^2 - Q_0^2) , \\ a^{(2)} &= \frac{C_F}{2} \left(\frac{67}{6} - \frac{\pi^2}{2} - \frac{5N_f}{9} \right) ,\end{aligned}\tag{42}$$

for (37), with two-loop α_s , $\Lambda_{QCD} = \Lambda_{\overline{MS}}$; and

$$M'(\tau W^2) = \frac{2C_F}{\pi} \int_{\tau}^1 dx \left(\frac{1}{x} - 1 + \frac{x}{2} \right) \alpha_s(x\tau W^2) M_g(x\tau W^2)\tag{43}$$

for (40)⁴.

The distribution $F(\tau)$ can be evaluated either directly (numerically) [14][15], or using Laplace transformation [19]. It is an infrared-safe quantity in the following sense: Though each term of the series (36) depends on the infrared cutoff Q_0 , the dependence disappears in the sum at the high energy limit. In fact, the result somewhat depends on the value of Q_0 at a finite energy, which gives rise to the ambiguity of the order of Q_0/W . We do not know theoretically the exact value of Q_0 , because it is associated with the non-perturbative hadronisation process.

In order to minimise the theoretical ambiguity, we use the experimentally observed thrust distribution for $F(\tau)$ on the rhs of (41). Detailed data in the small τ region was obtained by TOPAZ collaboration at TRISTAN at $W=58\text{GeV}$ [20]. The result of the numerical evaluation of (41) is shown in Fig.3. Note that the experimental data of $F(\tau)$ are given as the average values in finite bin sizes. Though the numerator of the rhs of (41) is continuous in τ , its denominator is not. In order to reduce the error, we evaluate (41) only at the middle value of τ in each bin, where the data of $F(\tau)$ is not far from its real value. In the integrand on the numerator, $F(\tau - \tau_1)$ is regarded as a step function (constant in each bin). In Fig.3, the result at each τ is connected one another by straight lines. The curve thus obtained is still not smooth because of the inaccuracy included in this procedure.

⁴The term proportional to $a^{(2)}$ in the gluon emission amplitude off a quark (see (42)) gives the next-to-next correction in evaluating $M'(\tau W^2)$. In fact, the leading-log approximation (36) is organised differently from MLLA, and we cannot tell a priori whether the next-to-next corrections in MLLA are in fact smaller than the next-to-leading order corrections in (36). (The former haven't been calculated yet.)

Let us now examine the validity of the approximation. First, we discuss the small τ region.

It has sometimes been argued that at the current energies the thrust distribution in the small τ region ($\tau < 0.07$ or so) was determined mostly by the hadronisation process. The argument was partly based on the observation in the Monte Carlo simulation studies: The thrust distribution at the parton level (with partons produced above certain transverse momentum cutoff of the order of 1GeV) is modified substantially in the small τ region by the hadronisation (i.e. with the phenomenological hadronisation models to convert the particles produced by the QCD parton shower into hadrons). If so, use of (36) for $F(\tau)$ (also (39) for $\langle n \rangle_\tau$) would not be justified in the small τ region.

A recent study [15], however, has shown that when we lower the cutoff in the partonic thrust distribution, the result reproduces the data quite well. It suggests either that the non-perturbative hadronisation process is almost irrelevant to the distribution, or that the non-perturbative hadroproduction can be simulated by the perturbative soft-gluon emission. Whichever the case, we may safely assume that the LPHD holds also in small τ -processes⁵.

Another question to be addressed is on the recoil effect. In deriving the formulae, we have neglected the fact that an emitted gluon carries out a part of the energy of the parent quark, and accordingly the succeeding emission makes a reduced contribution to τ (recoil effect) [15]. The recoil is large at the large τ region. At the small τ region, where soft-gluon emission dominates, the recoil is negligible. In fact, the correction at the large τ region modifies the distribution at a small τ through the change in its overall normalisation. The change in the normalisation of $F(\tau)$, however, is canceled on the rhs of (41) between the numerator and the denominator.

At large τ , on the other hand, (41) overestimates the multiplicity. The thrust distribution in the large τ -region is determined by the Feynmann diagrammes at the lowest orders in α_s . Because the $O(\alpha_s)$ matrix element vanishes for $\tau \geq 1/3$, the thrust distribution decreases rapidly above $\tau \approx 1/3$. The numerator on rhs of (41), on the other hand, does not vanish at $\tau \geq 1/3$ even if $F(\tau_1)$ vanishes for $\tau_1 \geq 1/3$ in its integrand. The overestimation is particularly large when τ is close to $1/3$.

Apart from this obvious overestimation near $\tau = 1/3$ (see Fig.3 at $\tau \approx 1/3$), the recoil effect causes the reduction of the multiplicity in a wider τ region. In addition to

⁵ The assumption that the LPHD is correct with a small infrared cutoff is consistent with the experiences in the particle spectrum[4].

the recoil effect on $F(\tau)$, already discussed above, the recoil from the emission of a hard gluon reduces the multiplicity from the quark-antiquark pair, which partly compensates the multiplicity increase from the hard gluon (see the discussion given at the end of Sect.2). This effect is not included in (41). With the recoil effect, the multiplicity increase in τ at large τ would be less than the one estimated by (41).

For the intermediate value of τ ($\tau \approx 0.1-0.2$), the overestimation reduces. On the other hand, we note that at very high energies the integrand in the numerator on the rhs of (41) has two sharp peaks, one at $\tau_1 \approx 0$ (near the lower bound) and the other at $\tau_1 \approx \tau$ (near the upper bound). The contribution from the former peak corresponds to the multiplicity in the quark-(or antiquark)-jet, while the contribution from the latter corresponds to the multiplicity in the gluon-jet. We may therefore expect that (41) would give a prediction close to (20) in the intermediate τ region.

In Fig.3, we also show $\langle n \rangle_\tau$ evaluated by (20) in order to compare it with (41). We find that (41) gives smaller multiplicity than (20) (except near $\tau = 1/3$), despite the fact that the neglect of the recoil effect tends to give an overestimation in (41) in the large and intermediate τ region. The reason for the difference is not difficult to find. At 58GeV, the thrust distribution is not very sharply peaked at very small τ , so that in the integration in the numerator on the rhs of (41) the peak corresponding to the multiplicity in the gluon jet does not develop enough to give a contribution separate from the multiplicity from the $q\bar{q}$ -pair. (The maximum of $d\sigma/d\ln(1/\tau)$ is at $\tau \approx 0.043$.) If the jets are not collimated enough, it may be difficult to determine the multiplicity ratio from the multiplicity data with a thrust cut at this energy.

Finally, let us make a comment on the determination of the normalisation of the multiplicity function M_g , which has been implicit so far. Because $F(\tau)$ is a normalised distribution, we should have

$$\int_0^{1/2} d\tau F(\tau) < n >_\tau = < n >_{e^+e^-} . \quad (44)$$

On the other hand, when we integrate $F(\tau)\langle n \rangle_\tau$ given by (41) over $0 < \tau < 1$, we obtain

$$\begin{aligned} \int_0^1 d\tau F(\tau) < n >_\tau &= \int_0^1 d\tau \int_0^\tau \frac{d\tau_1}{\tau_1} M'(\tau_1 W^2) F(\tau - \tau_1) \\ &= \int_0^1 d\tau \int_0^1 \frac{d\tau_1}{\tau_1} M'(\tau_1 W^2) \int_0^1 d\tau_2 F(\tau_2) \delta(\tau - \tau_1 - \tau_2) \\ &= \int_0^1 \frac{d\tau_1}{\tau_1} M'(\tau_1 W^2) \int_0^1 d\tau_2 F(\tau_2) \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \frac{d\tau_1}{\tau_1} M'(\tau_1 W^2) \\
&= \langle n \rangle_{e^+e^-} \quad .
\end{aligned} \tag{45}$$

The final equality in (45) is correct in the accuracy of the next-to-leading order (see (34) in the last section). Indeed, the value of τ cannot exceed $1/2$, and therefore the condition (44) is not satisfied if the multiplicity function M_g is so normalised that the next-to-leading order formula for $\langle n \rangle_{e^+e^-}$ (34) be satisfied. Alternatively, we may normalised the multiplicity function M_g so that the condition (44) be satisfied. We used the latter normalisation when we obtained the curve shown in Fig.3.

The condition (44), however, does not completely remove the error in the normalisation. As is discussed above, the factorisation formula (41) gives overestimations of the multiplicity in the large τ region. If the multiplicity is normalised by the use of (44), therefore, the multiplicity is underestimated in the small τ region. When we compare the prediction with the experimental data in the small τ region, the error in the normalisation (of the order of α_s) should be taken into account.

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W(GeV)	$\tau=0.1$	$\tau=0.2$	$\tau=0.3$
91	1.33	1.53	1.60
58	1.31	1.51	1.58

Table 1 A comparison of the predictions of the ratio $\langle n \rangle_\tau / \langle n \rangle_{e^+e^-}$ at different energies.

FIGURE CAPTIONS

Fig.1 A schematic representation of the one-particle-inclusive cross section at a given τ . The dashed line represents the registered particle. Successive soft-gluon emission takes place in the shaded blob.

- (a) The registered particle originates from the hard quark.
- (b) The registered particle originates from the hard gluon.

Fig.2 The prediction for $(\langle n \rangle_g + \langle n \rangle_{q\bar{q}}) / \langle n \rangle_{e^+e^-}$ (solid curve), and the multiplicity ratio $R(\tau) = \langle n \rangle_g / \langle n \rangle_q$ (dashed curve). $W = 58\text{GeV}$ and $\Lambda_{QCD} = 0.15\text{GeV}$.

Fig.3 The predictions for $\langle n \rangle_\tau / \langle n \rangle_{e^+e^-}$ ($W = 58\text{GeV}$): The approximation (41) (solid curve) vs the approximation (20) (dashed curve).

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